1

Let A_{ij} be the topology described in the ith row and jth column of **ex1**.

A_{11} is the indiscrete topology, is comparable with all others, and is coarser than all others.

A_{33} is the discrete topology, is comparabale with all others, and is finer than the rest.

A_{12} is finer than A_{31} and coarser than A_{32}.

A_{13} is finer than A_{31} and A_{21}, and coarser than A_{23}.

A_{21} is coarser than A_{23} and A_{32}.

A_{23} is finer than A_{31}.

A_{31} is coarser than A_{32}.

Any relationship not explicitly listed as coarser or finer is incomparable.

2

(a1) Let $\{\tau_\alpha\}$ be a family of topologies on X, and let $\cap\tau_\alpha = \tau$.

Clearly, $\emptyset \in \tau$ and $X \in \tau$.

Let $y, z \in \tau$. Then $y, z \in \tau_\alpha$ for all $\tau_\alpha \in \{\tau_\alpha\}$. So by definition, $y \cap z \in \tau_\alpha$, so $y \cap z \in \tau$.

Suppose $A \subseteq \mathbb{P}(\tau)$. Then A is a subset of each τ_α. Then by definition, $\cup_\lambda A_\lambda \in \tau_\alpha$, so $\cup_\lambda A_\lambda \in \tau$.

As τ is closed under pairwise intersection and arbitrary union, and includes X and $\emptyset$, τ is a topology on X.

(a2) Suppose, rather, $\tau = \cup\tau_\alpha$. Consider $X = \{a, b\}$, $\tau_1 = \{\emptyset, \{a\}, X\}$, and $\tau_2 = \{\emptyset, \{b\}, X\}$. Then $\{a\} \in \tau$ and $\{b\} \in \tau$, but $\{a\} \cup \{b\} = \{a, b\} \notin \tau$.

(b1) Let $\{\tau_\alpha\}$ be a family of topologies on X and $S = \cup_\alpha \tau_\alpha$. Clearly S covers X, as at least one (indeed, every) τ_α contains X. Thus, S is a subbasis. Let β then be the set of all finite intersections of elements of S, which certainly a basis, and let τ be the topology generated from β in the usual fasion. We see that $\tau \supseteq S$, by construction.

Suppose there is some topology γ such that $\gamma \supseteq S$. By definition of topology, all finite intersections of elements of γ are in γ. As $S \subseteq \gamma$ by construction, this implies that $\beta \subseteq \gamma$. However, we also know that arbitrary unions of members of γ are in γ. As $\beta \subseteq \gamma$, this implies $\tau \subseteq \gamma$. Therefore, any topology containing S must contain τ. So γ either is τ, or is finer than τ.

This proves τ is a unique smallest superset of S.

(b2) I'm reading the question as "the largest topology contained in *each* τ_α", since otherwise it seems very situational.

Let $\{\tau_\alpha\}$ be as above, and let $\tau = \cap_\alpha \tau_\alpha$. Suppose γ is a topology of X such that $\gamma \subseteq \tau_\alpha$ for each τ_α. Then clearly, $\gamma \subseteq \tau$. Thus, either $\tau = \gamma$ or $\tau \supset \gamma$. So τ is the unique largest set contained in each τ_α.

(c1)

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, X\}$$

(c2)

$$\{\emptyset, \{a\}, X\}$$

3

Let A be a basis for some topology on X, $\{T_\alpha\}$ be the set of topologies on X containing A, $T = \cap_\alpha T_\alpha$, and $\tau = \tau_A$.

(a) We will show that $\tau_A = T$. First, we will show that $\tau \subseteq T$. Let $\gamma \in \tau$. By construction, there exists some family of subsets $\psi \subseteq A$ such that $\cup\psi = \gamma$. Then as $A \subseteq T_\alpha$ (also by construction), $\psi \subseteq T_\alpha$ for every T_α. Then as each T_α is a topology, $\cup\psi \in T_\alpha$, so $\gamma \in T$. Thus, $\tau \subseteq T$.

As $\tau \supseteq A$, $\tau \in \{T_\alpha\}$. Thus, for any $t \in T$, $t \in \tau$ by construction. So $T \subseteq \tau$.

Therefore, $T = \tau$.

(b) Suppose that A is merely a subbasis. Then β is the basis constructed from A in the usual manner, and likewise $\tau = \tau_\beta$. As each T_α contains A, each must also contain all finite intersections of A, by the definition of topology. So $\beta \subseteq T$. As T is a topology by **4a**, all arbitrary unions of β are members of T, so $\tau \subseteq T$.

As $\tau \supseteq A$, $\tau \in \{T_\alpha\}$. Thus, $\tau \supseteq T$.

Therefore, $T = \tau$.

4

(a) Let $\beta = \{(a,b) : a < b, a, b \in \mathbb{Q}\}$. Suppose (x,y) is some arbitrary member of the standard basis on $\mathbb{R}$. Define a sequence (x_n) as follows: x_1 is a rational number in (x,y), which is guarenteed to exist as the rationals are dense in $\mathbb{R}$, and x_n is a rational number in $(x, x + \frac{x_{n-1}-x}{2})$. Define (y_n) similarly. Each $(x_n, y_n) \in \beta$, so $\cup_n (x_n, y_n) = U \in \tau_\beta$.

Let $\alpha \in (x,y)$, and $\epsilon = \alpha - x$, which is certainly some positive real. As $x_n \to x$, there is some $j \in \mathbb{N}$ such that $|x_j - x| < \epsilon$, meaning $\alpha > x_j$. A similar argument gives some $y_k > \alpha$. With $n = \max(j,k)$, $\alpha \in (x_n, y_n)$. Thus, $(x,y) \subseteq U$.

Let $\alpha \notin (x, y)$. Without loss of generality, assume $\alpha \leq x$. Then for all x_n, $x_n > \alpha$. Thus, $\alpha \notin U$.

As $(x, y) \subseteq U$ and $(x, y)^C \subseteq U^C$, $(x, y) = U$. So τ_β contains the standard basis on $\mathbb{R}$. Of course, since topologies are closed under arbitrary union, this implies that $\tau_\beta \supseteq \tau_{\mathbb{R}}$.

Since $\beta_{\mathbb{R}} \supseteq \beta$, we already have $\tau_\beta \subseteq \tau_{\mathbb{R}}$.

Therefore, $\tau_\beta = \tau_{\mathbb{R}}$.

(b) Let $\kappa = \{[a, b) : a < b, a, b \in \mathbb{Q}\}$.

First we show that κ is a basis. Let $x \in \mathbb{R}$. Then there are certainly rationals p and q such that $x - 1 < p < x < q < x + 1$, as $\mathbb{Q}$ is dense in $\mathbb{R}$. Then $x \in [p, q)$.

Let $[p, q), [g, h) \in \kappa$, and let $I = [p, q) \cap [g, h)$. Assume without loss of generality that $h > q$. If $h \geq q$, then $I = \emptyset$. If $p \leq h < q$, then $I = [h, q)$. If $h < p$, then $I = [p, q)$. In the first case, there can be no $x \in I$, so basis conditions are trivially satisfied. In the remainder, $I \in \kappa$, so basis conditions are satisfied.

Consider $[\pi, 4)$. This is a member of $\mathbb{R}_\ell$. But no union or finite intersection of members of κ can result in a set which includes π but no values less than π. Therefore $\mathbb{R}_\ell$ is a different topology from $\mathbb{R}_\kappa$.